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This Symposium has dealt with provable algorithms for finding zeros of general polynomials, with the tacit assumption that the processes would be implemented on an ideal computer system capable of exact arithmetic operations. In contrast, I should like to point out the near absence of algorithms to solve even a quadratic equation in a satisfactory way on actually used digital computer systems. The difficulties are partly caused by round-off error in floating-point arithmetic, but much more by the ever-present possibility of overflow or underflow (defined below).

This note presents specifications for a satisfactory quadratic equation solver suggested by Professor W. Kahan of the University of Toronto in lectures at Stanford University in 1966. The general level of performance is Kahan's, but the details are mine.

Consider the following set $F = F(\beta, t, m, M)$ of normalized floating-point numbers. This uses a number base β (bases 2, 8, 10, and 16 are in use) and a prescribed number t of significant digits. There are two limiting integer exponents m and M . The set F contains precisely

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$$(1) \quad 1 + 2(\beta - 1)(M - m + 1)\beta^{t-1}$$

numbers. One of these is 0. Each other number in F has a unique representation

$$(2) \quad \pm N \times \beta^e,$$

where the sign is + or -, where the integer exponent e satisfies the inequality

$$(3) \quad m \leq e \leq M,$$

and where the integer significand N satisfies the normalization condition

$$(4) \quad \beta^{t-1} \leq N \leq \beta^t - 1.$$

Frequently 0 is given the unique computer representation

$$(5) \quad + 0 \times \beta^m.$$

See [1] for a discussion of this floating-point number system in a slightly different notation.

We choose F^* to be a certain subset of F consisting of numbers not too close to overflow or underflow.

Definition. To be definite (and somewhat arbitrary), let $F^* = F^*(\beta, t, m, M)$ be the set consisting of 0 and all numbers (2)

subject to (4) and also to

$$(3^*) \quad m + 1 \leq e \leq M - 1 .$$

Definitions. A real number x is said to be in the range of F^* if either $x = 0$ or

$$(6) \quad \beta^{t-1} \beta^{m+1} \leq |x| \leq (\beta^t - 1) \beta^{M-1} .$$

A complex number z is in the range of F^* if both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers in the range of F^* .

One similarly defines the expression in the range of F for real and complex numbers.

The statement (in computer jargon) that a real number y suffers from over- or underflow is equivalent to saying that y is not in the range of F .

In terms of these concepts I now give specifications in the form of a commented heading in Algol 60 of what I consider to be a satisfactory quadratic-equation solver for use with a processor of floating-point numbers in $F(\beta, t, m, M)$.

procedure QUADRATIC ($a, b, c, x1, y1, x2, y2, \text{error}$);

value a, b, c ; real $a, b, c, x1, y1, x2, y2$; switch error ;

comment We are solving the equation $az^2 + bz + c = 0$, for arbitrary input parameters a, b, c in F^* . Where values of the output parameters are not specified, they are irrelevant.

If $a = b = c = 0$, exit to error[1], since all complex numbers z satisfy the equation.

If $a = b = 0$ and $c \neq 0$, exit to error[2], since no z satisfies the equation.

Otherwise, let z_1 and z_2 be the exact roots of the equation, numbered so that $|z_1| \leq |z_2|$. (If $a = 0$, let $z_2 = \infty$.)

Whenever z_1 is in the range of F^* , set x_1 and y_1 to numbers in F close (defined below) to the real resp. imaginary part of z_1 .

Whenever z_2 is in the range of F^* , set x_2 and y_2 to numbers in F close to the real resp. imaginary part of z_2 .

Let $\zeta_1 = x_1 + i \times (y_1)$. We require that $\zeta_1 = 0$ (if $z_1 = 0$), and otherwise (again being somewhat arbitrary) that $|\zeta_1 - z_1| \leq \beta + 1$ units in the least-significant digit of the significand of the floating-point representation (2) of $\max(|x_1|, |y_1|)$. To repeat this requirement in symbols, if $\text{entier}[x]$ denotes the greatest integer $\leq x$, we demand that

$$(7) \quad |\zeta_1 - z_1| \leq (\beta + 1) \times \beta^e,$$

where $e = \text{entier}[\log_{\beta} \max(|x_1|, |y_1|) - t]$.

We make a corresponding requirement of $\zeta_2 = x_2 + i \times (y_2)$.

If z_1 is not in the range of F , but z_2 is in the range of F^* , set x_2, y_2 as above and exit to error[3].

If z_2 is not in the range of F (including the case $z_2 = \infty$), but z_1 is in the range of F^* , set x_1, y_1 as above and exit to error[4].

If neither z_1 nor z_2 is in the range of F , exit to error[5] .

If a root z_i is in the range of F but not in the range of F^* , we permit either an indication of over- or underflow via the appropriate exit to error, or a determination of z_i with an accuracy satisfying (7) above.

The procedure QUADRATIC should make no unnecessary use of multiple-precision computation, but computation with 2t significant digits is essential at one part of the procedure, to achieve the accuracy (7). End of comment;

The main source of practical difficulty in writing the procedure QUADRATIC is the possibility of over- or underflow in many places. What is actually programmed depends crucially on what the computing system does in case of over- or underflow. An ideal system permits the user's program to regain control of the algorithm, if the user wishes, with the ability to interrogate Boolean variables to learn whether there has been overflow, underflow, or neither. Though such systems are rare, they have been implemented at Toronto [3] and at Stanford [2], and these systems make programming such an algorithm as QUADRATIC far more satisfactory.

Some systems merely dismiss a user's program in case of overflow. If a result underflows, many systems merely set the result to 0 and return to the user's program without any indication. Faced with systems like these, the programmer must take great pains to insure that over- or underflow can never occur. These precautions make a satisfactory algorithm tedious to write, lengthy to store, and slow to execute. One necessary subroutine must determine the exponents of a , b , c , and other real

numbers local to the procedure. This is probably most gracefully written in machine code.

The ability to achieve the prescribed accuracy (7) of $\beta + 1$ units in the last place depends on the detailed properties of the floating-point arithmetic processor. If the prescribed accuracy is not achievable, condition (7) must be relaxed as necessary. If necessary, one could also make the set F^* smaller by changing (3^{*}).

As a simple illustration suppose that $\beta = 10$, $t = 4$, $m = -54$, $M = 45$. Then the smallest and largest positive numbers in F are

$$1000 \times 10^{-54} = 10^{-51}$$

and

$$9999 \times 10^{45} = .9999 \times 10^{49} = (1 - 10^{-4}) \times 10^{49}.$$

Here are some equations that may give trouble for this system:

(a) $10^{-40} z^2 - 5 \times 10^{-40} z + 6 \times 10^{-40} = 0:$

The roots are 2 and 3, and the only danger is that undetected underflows will introduce an error.

(b) $z^2 + 10^{10} z - 1 = 0:$

Use of the standard quadratic formula will yield 0 for the positive root, instead of the correctly rounded value 1.000×10^{-10} .

(c) $10^{-30} z^2 - 10^{40} + 1 = 0:$

The roots are near 10^{70} and 10^{-40} ; use of the quadratic formula can easily cause overflow or underflow.

$$(d) \quad 2.864 z^2 - 2.864 z + 0.7160 = 0:$$

Here 0.5000 is a double root. Evaluating the quadratic formula

$$\frac{2.864 \pm \sqrt{(2.864)^2 - (4 \times 2.864) \times 0.7160}}{2 \times 2.864}$$

in single-precision rounded arithmetic yields $0.5000 \pm 0.05477 i$, with an error of over 547 units in the last place of 0.5000.

It is not purely academic to make strict demands of the procedure QUADRATIC. Quadratic equations arising in the course of solving determinantal equations by Muller's method [6] or Laguerre's method [5], particularly in connection with large matrices, sometimes have one of the roots out of the range of F^* , and yet make essential use of the root in the range of F^* . The accuracy requirement (7) is perhaps overstrict for equations with nearly double roots.

I believe the specifications to be very reasonable for a basic process like solving a quadratic equation. Nevertheless, I venture to guess that not more than five quadratic solvers exist anywhere that meet the general level of the specifications. Kahan [4] has prepared an algorithm (in Fortran IV for the 7094-II under the Toronto version of the IBSYS operating system) which achieves the specifications for $\beta = 2$, $t = 27$, $m = -155$, $M = 100$. The error never exceeds $9/4 (= \beta + 1/4)$ units in the least-significant place of the root. The only multiple-precision operations occur in the computation of $\Delta = b_1^2 - 4a_1c_1$, followed by storage of a single-precision value of Δ . (Here a_1, b_1, c_1

are proportional to a, b, c .) The excess time for the double-precision computation is negligible in comparison with the time required to deal with over- and underflow.

It is obviously relevant to ask to what extent the various computer algorithms presented at this Symposium for general polynomials make provision for over- and underflow, and also what accuracy they achieve.

It is noteworthy that the programming of QUADRATIC depends crucially on the arithmetic properties of the computing system, especially on its behavior with over- and underflow. The practical numerical analyst with high standards is thus inextricably involved with the arithmetic behavior of his digital computer hardware and accompanying operating systems. Unfortunately, few numerical analysts have formulated their systems requirements explicitly, not to mention communicating them effectively to the persons who design hardware and software systems. With existing computing systems a numerical analyst faces a most disagreeable dilemma-- either he writes less than satisfactory algorithms, or he undertakes the never-ending chore of writing basic software systems (and perhaps even rebuilds the arithmetic unit). Professor Kahan has generally taken the second alternative.

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